SPACES OF SL(2)-ORBITS IN MIXED CASE : Joint work with Kazuya Kato and Chikara Nakayama (Hodge theory and algebraic geometry)

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INTRODUCTION

For a Griffiths domain D in [G], basing on the SL(2)-orbit theorem of Cattani, Kaplan and Schmid [CKS], Kato and the author constructed in [KU2] the space of SL(2)-orbits $D_{\mathrm{SL}(2)}$ which is an enlargement of D.

We evolve this to the mixed case. Let now D be a classifying space of mixed Hodge structures with polarized graded quotients defined in [U]. The construction of our spaces of SL(2)-orbits are now based on a mixed Hodge theoretic version of SL(2)-orbit theorem obtained by Kato, Nakayama and the author in [KNU1]. Let $D_{\mathrm{SL}(2)}$ be the set of torus orbits associated to SL(2)-orbits in mixed case (for the precise definition, see §5). We define (in §7) two structures on the set $D_{\mathrm{SL}(2)}$ as real analytic manifold endowed with a log structure with sign, allowing corners and slits (i.e., as object of $\mathcal{B}_{\mathbf{R}}(\log)$ in §6), which contains D as a dense open subset. We denote $D_{\mathrm{SL}(2)}$ with these structures by $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$. There is a morphism $D^{I}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ whose underlying map is the identity map of $D_{\mathrm{SL}(2)}$. The log structure with sign of $D^{I}_{\mathrm{SL}(2)}$ coincides with the inverse image of that of $D^{II}_{\mathrm{SL}(2)}$.

In the pure case, these two structures coincide, and the topology of $D_{\mathrm{SL}(2)}$ given by these structures coincides with the one defined in [KU2].

$D^{II}_{\mathrm{SL}(2)}$ is proper over $\mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ (7.9). This shows that our definition of $D_{\mathrm{SL}(2)}$ in the mixed case provides sufficiently many points at infinity. This properness is a good property of $D^{II}_{\mathrm{SL}(2)}$ which $D^{I}_{\mathrm{SL}(2)}$ need not have. On the other hand, $D^{I}_{\mathrm{SL}(2)}$ is nice for norm estimate (§8), but $D^{II}_{\mathrm{SL}(2)}$ need not be. The topologies of $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ often differ.

In Part I ([KNU2]) of a series of papers "Classifying spaces of degenerating mixed Hodge structures", joint with Kato and Nakayama, we constructed and studied Borel-Serre space D_{BS} which is a real analytic manifold with corners like original Borel-Serre space in [BS]. In Part II ([KNU5]), we constructed and studied spaces of SL(2)-orbits $D_{\mathrm{SL}(2)}$. This is the topic of the present note. The spaces of nilpotent orbits D_{Σ} are constructed and studied in Part III ([KNU6]). These spaces D_{Σ}, $D_{\mathrm{SL}(2)}$, and D_{BS} belong

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to the following fundamental diagram of eight enlargements of D whose construction is one aim of this series of papers.

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ D_{\Sigma,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\ \downarrow & & \downarrow & & & & \\ D_{\Sigma} & \leftarrow & D^{\sharp}_{\Sigma} & & & & \end{array}$$

This fundamental diagram in the pure case was constructed in [KU3].

This is a note for the conference talk, which is edited a joint work [KNU5] to make a brief guide. [KNU5] contains a theory and applications of log modification for log structures with sign. This is also an important topic, but we omit it here. The author takes full responsibility for the wording and content of this note.

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§1. Classifying space

We recall standard notation.

Fix a quadruple $(H_0, W, (\langle\ ,\ \rangle_w)_{w\in\mathbf{Z}}, (h^{p,q})_{p,q\in\mathbf{Z}})$, where

H_0 : a finitely generated free $\mathbf{Z}$-module,
W : an increasing filtration on $H_{0,\mathbf{R}} := \mathbf{R}\otimes_{\mathbf{Z}} H_0$ defined over $\mathbf{Q}$,
$\langle\ ,\ \rangle_w$: a non-degenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ defined over $\mathbf{Q}$ for each $w \in \mathbf{Z}$ which is symmetric if w is even and anti-symmetric if w is odd,
$h^{p,q}$: a non-negative integer given for $p, q \in \mathbf{Z}$ such that $h^{p,q} = h^{q,p}$, $\mathrm{rank}_{\mathbf{Z}}(H_0) = \sum_{p,q} h^{p,q}$, and $\dim_{\mathbf{R}}(\mathrm{gr}^W_w) = \sum_{p+q=w} h^{p,q}$ for all w.

Let $\check{D}$ be the set of all decreasing filtrations F on $H_{0,\mathbf{C}} := \mathbf{C}\otimes_{\mathbf{Z}} H_0$ satisfying the following two conditions.

(1) $\dim(F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q})) = h^{p,q}$ $(\forall p, q \in \mathbf{Z})$.

(2) $\langle\ ,\ \rangle_w$ kills $F^p(\mathrm{gr}^W_w) \times F^q(\mathrm{gr}^W_w)$ $(\forall p, q, w \in \mathbf{Z}$ such that $p + q > w)$.

Here $F(\mathrm{gr}^W_w)$ denotes the filtration on $\mathrm{gr}^W_{w,\mathbf{C}} := \mathbf{C}\otimes_{\mathbf{R}} \mathrm{gr}^W_w$ induced by F.

Let D be the set of all decreasing filtrations $F \in \check{D}$ which also satisfy the following condition.

(3) $i^{p-q}\langle x, \bar{x}\rangle_w > 0$ $(0 \neq \forall x \in F^p(\mathrm{gr}^W_w) \cap \overline{F^q(\mathrm{gr}^W_w)}$, $\forall p, q, w \in \mathbf{Z}$ with $p + q = w)$.

Then, D is an open subset of $\check{D}$ and, for each $F \in D$ and $w \in \mathbf{Z}$, $F(\mathrm{gr}^W_w)$ is a Hodge structure on $(H_0 \cap W_w)/(H_0 \cap W_{w-1})$ of weight w with Hodge number $(h^{p,q})_{p+q=w}$ which is polarized by $\langle\ ,\ \rangle_w$. The space D is the classifying space of mixed Hodge structures of type $(H_0, W, (\langle\ ,\ \rangle_w)_{w\in\mathbf{Z}}, (h^{p,q})_{p,q\in\mathbf{Z}})$ in [U], which is a natural generalization of Griffiths domain in [G] to the mixed case. These two are related by taking graded quotients by W as follows.

$D(\mathrm{gr}^W_w)$: the D for $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w, (h^{p,q})_{p+q=w})$ $(\forall w \in \mathbf{Z})$.

$D(\mathrm{gr}^W) = \prod_{w\in\mathbf{Z}} D(\mathrm{gr}^W_w)$.

$D \to D(\mathrm{gr}^W)$, $F \mapsto F(\mathrm{gr}^W) := (F(\mathrm{gr}^W_w))_{w\in\mathbf{Z}}$, the canonical surjection.

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$,

G_A: the group of all A-automorphisms g of $H_{0,A} := A \otimes_{\mathbf{Z}} H_0$ compatible with W such that $\mathrm{gr}^W_w(g) : \mathrm{gr}^W_w \to \mathrm{gr}^W_w$ are compatible with $\langle\ ,\ \rangle_w$ for all w.

$G_{A,u} := \{g \in G_A \mid \mathrm{gr}^W_w(g) = 1 \text{ for all } w \in \mathbf{Z}\}$, the *unipotent radical* of G_A.

$G_A(\mathrm{gr}^W_w)$: the G_A of $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w)$ for each $w \in \mathbf{Z}$.

$G_A(\mathrm{gr}^W) := \prod_w G_A(\mathrm{gr}^W_w)$.

Then, $G_A/G_{A,u} = G_A(\mathrm{gr}^W)$, and G_A is a semi-direct product of $G_{A,u}$ and $G_A(\mathrm{gr}^W)$.

The natural action of $G_{\mathbf{C}}$ on $\check{D}$ is transitive, and $\check{D}$ is a complex homogeneous space under this action. Hence $\check{D}$ is a complex analytic manifold. An open subset D of $\check{D}$ is also a complex analytic manifold. However, the action of $G_{\mathbf{R}}$ on D is not transitive in general (see the equivalent conditions (4), (5) below). The subgroup $G_{\mathbf{R}}G_{\mathbf{C},u}$ of $G_{\mathbf{C}}$ acts always transitively on D, and the action of $G_{\mathbf{C},u}$ on each fiber of $D \to D(\mathrm{gr}^W)$ is transitive.

$\mathrm{spl}(W)$: the set of all isomorphisms $s : \mathrm{gr}^W = \bigoplus_w \mathrm{gr}^W_w \xrightarrow{\sim} H_{0,\mathbf{R}}$ of $\mathbf{R}$-vector spaces such that for any $w \in \mathbf{Z}$ and $v \in \mathrm{gr}^W_w$, $s(v) \in W_w$ and $v = (s(v) \bmod W_{w-1})$.

We have the action $G_{\mathbf{R},u} \times \mathrm{spl}(W) \to \mathrm{spl}(W)$, $(g, s) \mapsto gs$.

For a fixed $s \in \mathrm{spl}(W)$, we have a bijection $G_{\mathbf{R},u} \xrightarrow{\sim} \mathrm{spl}(W)$, $g \mapsto gs$. Via this bijection, we endow $\mathrm{spl}(W)$ with a structure of a real analytic manifold.

$D_{\mathrm{spl}} := \{s(F) \mid s \in \mathrm{spl}(W),\ F \in D(\mathrm{gr}^W)\} \subset D$, the subset of $\mathbf{R}$-*split* elements.

Here $s(F)^p := s(\bigoplus_w F^p_{(w)})$ for $F = (F_{(w)})_w \in D(\mathrm{gr}^W)$.

$D_{\mathrm{nspl}} := D \smallsetminus D_{\mathrm{spl}}$.

Then, D_{spl} is a closed real analytic submanifold of D, and we have a real analytic isomorphism $\mathrm{spl}(W) \times D(\mathrm{gr}^W) \xrightarrow{\sim} D_{\mathrm{spl}}$, $(s, F) \mapsto s(F)$.

The following two conditions are equivalent ([KNU2], Proposition 8.7).

(4) D is $G_{\mathbf{R}}$-homogeneous.

(5) $D = D_{\mathrm{spl}}$.

For example, if there is $w \in \mathbf{Z}$ such that $W_w = H_{0,\mathbf{R}}$ and $W_{w-2} = 0$, then the above equivalent conditions are satisfied. But in general these conditions are not satisfied (see Examples I, III, IV below).

Examples. Let E and E' be elliptic curves.

	gr^W	geometry
0	$H^1(E)(1)$	period of E
I	$\mathbf{Z}(1) \oplus \mathbf{Z}$	extension data: $\mathbf{Z}\backslash\mathbf{C} \simeq \mathbf{C}^*$
II	$H^1(E)(1) \oplus \mathbf{Z}$	$J(H^1(E)) = E$
III	$H^1(E)(2) \oplus \mathbf{Z}$	regulator
IV	$\mathbf{Z}(1) \oplus H^1(E)(1) \oplus \mathbf{Z}$	$E \leftarrow$ (Poincaré bundle) $\to$ (dual of E)
V_1	$\mathrm{Sym}^3(H^1(E))(2) \oplus \mathbf{Z}$	intermediate Jacobian of CY with $h^{2,1} = 1$
V_2	$H^1(E)(1) \oplus \mathrm{Sym}^2(H^1(E'))(1)$	
V_3	$\mathrm{Sym}^2(H^1(E'))(1) \oplus H^1(E)(1)$	$H^2((\text{K3 with } \rho = 19) \smallsetminus (\text{elliptic curve}))(1)$

For the precise description of the spaces D with the same type of graded quotients as these examples, see §9 below.

§2. Decomposition of D as real analytic manifold

Let W and D be as in §1. In this section, we review the canonical splitting $\mathrm{spl}_W(F') \in \mathrm{spl}(W)$ of the weight filtration W associated to $F' \in D$, defined by the theory of Cattani-Kaplan-Schmid [CKS]. In §1 of [KNU1] reviewed the definition of $\mathrm{spl}_W(F')$ in detail.

Let $F = (F_{(w)})_w \in D(\mathrm{gr}^W)$. Regard F as the filtration $\bigoplus_w F_{(w)}$ on $\mathrm{gr}^W_{\mathbf{C}} = \bigoplus_w \mathrm{gr}^W_{w,\mathbf{C}}$, and let $H_F^{p,q} = H_{F_{(p+q)}}^{p,q} \subset \mathrm{gr}^W_{p+q,\mathbf{C}}$. Let

$$\mathcal{L} = \mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)_{\leq -2}$$

be the set of all $\mathbf{R}$-linear maps $\delta : \mathrm{gr}^W \to \mathrm{gr}^W$ such that $\delta(\mathrm{gr}^W_w) \subset \bigoplus_{w' \leq w-2} \mathrm{gr}^W_{w'}$ for any $w \in \mathbf{Z}$. Let

$$\mathcal{L}(F) = \{\delta \in \mathcal{L} \mid \delta(H_F^{p,q}) \subset \textstyle\bigoplus_{p'<p,q'<q} H_F^{p',q'} \text{ for all } p, q \in \mathbf{Z}\}.$$

Then, a result in [CKS] is reformulated as follows.

Theorem. *We have an isomorphism of real analytic manifolds*

$$D \simeq \{(s, F, \delta) \in \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times \mathcal{L} \mid \delta \in \mathcal{L}(F)\}, \quad F' \mapsto (\mathrm{spl}_W(F'), F'(\mathrm{gr}^W), \delta(F')),$$

whose inverse is given by $(s, F, \delta) \mapsto s(\theta(F, \delta))$.

We explain the above correspondences.

For $F' \in D$, there is a unique pair $(s', \delta) \in \mathrm{spl}(W) \times \mathcal{L}(F'(\mathrm{gr}^W))$ such that

$$F' = s'(\exp(i\delta)F'(\mathrm{gr}^W))$$

([CKS]). This is the definition of $\delta = \delta(F')$ associated to F'.

Let $F' \in D$, and let $s' \in \mathrm{spl}(W)$ and $\delta \in \mathcal{L}(F'(\mathrm{gr}^W))$ be as above. Then the canonical splitting $s = \mathrm{spl}_W(F')$ of W associated to F' is defined by

$$s = s' \exp(\zeta),$$

where $\zeta = \zeta(F'(\mathrm{gr}^W), \delta)$ is a certain element of $\mathcal{L}(F'(\mathrm{gr}^W))$ determined by $F'(\mathrm{gr}^W)$ and δ in the following way.

Let $\delta_{p,q}$ $(p, q \in \mathbf{Z})$ be the (p, q)-Hodge component of δ with respect to $F'(\mathrm{gr}^W)$ defined by

$$\delta = \textstyle\sum_{p,q} \delta_{p,q} \quad (\delta_{p,q} \in \mathcal{L}_{\mathbf{C}}(F'(\mathrm{gr}^W)) = \mathbf{C} \otimes_{\mathbf{R}} \mathcal{L}(F'(\mathrm{gr}^W))),$$

$$\delta_{p,q}(H^{k,l}_{F'(\mathrm{gr}^W)}) \subset H^{k+p,l+q}_{F'(\mathrm{gr}^W)} \qquad \text{for all } k, l \in \mathbf{Z}.$$

Then the (p, q)-Hodge component $\zeta_{p,q}$ of $\zeta = \zeta(F'(\mathrm{gr}^W), \delta)$ with respect to $F'(\mathrm{gr}^W)$ is given as a certain universal Lie polynomial of $\delta_{p',q'}$ $(p', q' \in \mathbf{Z},\ p' \le -1,\ q' \le -1)$. See [CKS] and Section 1 of [KNU1]. For example,

$$\begin{aligned} \zeta_{-1,-1} &= 0, \\ \zeta_{-1,-2} &= -\tfrac{i}{2}\delta_{-1,-2}, \\ \zeta_{-2,-1} &= \tfrac{i}{2}\delta_{-2,-1}. \end{aligned}$$

For $F \in D(\mathrm{gr}^W)$ and $\delta \in \mathcal{L}(F)$, we define a filtration $\theta(F, \delta)$ on $\mathrm{gr}^W_{\mathbf{C}}$ by

$$\theta(F, \delta) = \exp(-\zeta) \exp(i\delta) F,$$

where $\zeta = \zeta(F, \delta)$ is the element of $\mathcal{L}(F)$ associated to the pair (F, δ) as above.

For $g = (g_w)_w \in G_{\mathbf{R}}(\mathrm{gr}^W) = \prod_w G_{\mathbf{R}}(\mathrm{gr}^W_w)$, we have

$$g\theta(F, \delta) = \theta(gF, \mathrm{Ad}(g)\delta),$$

where $\mathrm{Ad}(g)\delta = g\delta g^{-1}$.

For $F \in D(\mathrm{gr}^W)$, $\delta \in \mathcal{L}(F)$ and $s \in \mathrm{spl}(W)$, the element $s(\theta(F,\delta))$ of D belongs to D_{spl} if and only if $\delta = 0$.

We consider Examples 0–IV, V_3 in §1. For these examples, $\mathcal{L}(F) \subset \mathcal{L}$ is independent of the choice of $F \in D(\mathrm{gr}^W)$, which is denoted by L. By the above Theorem, we have a real analytic presentation of D

$$D \simeq \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times L.$$

The following is the table of correspondence with the complex analytic presentations.

	D	$\mathrm{spl}(W) \times D(\mathrm{gr}^W) \times L$
0	$\mathfrak{h}$	$\{0\} \times \mathfrak{h} \times \{0\}$
I	$\mathbf{C} \oplus \mathbf{Z}$	$\mathbf{R} \times \{\mathrm{pt}\} \times \mathbf{R}$
II	$\mathfrak{h} \times \mathbf{C}$	$\mathbf{R}^2 \times \mathfrak{h} \times \{0\}$
III	$\mathfrak{h} \times \mathbf{C}^2$	$\mathbf{R}^2 \times \mathfrak{h} \times \mathbf{R}^2$
IV	$\mathfrak{h} \times \mathbf{C}^3$	$\mathbf{R}^5 \times \mathfrak{h} \times \mathbf{R}$
V_3	$\mathfrak{h}^{\pm} \times \mathfrak{h} \times \mathbf{C}^3$	$\mathbf{R}^6 \times (\mathfrak{h}^{\pm} \times \mathfrak{h}) \times \{0\}$

Here $\mathfrak{h}^{\pm}$ is the disjoint union of $\mathfrak{h}^{+} = \mathfrak{h}$ and $\mathfrak{h}^{-} = \{x + iy \mid x \in \mathbf{R},\, 0 > y > -\infty\}$ ($\mathfrak{h}^{+} \simeq \mathfrak{h}^{-}$, $x + iy \mapsto x - iy$). For more precise description, see §9 below.

§3. Nilpotent orbits and SL(2)-orbits in mixed case

3.1. Let $N_j \in \mathfrak{g}_{\mathbf{R}}$ $(1 \le j \le n)$ and let $F \in \check{D}$. We say $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit* if the following conditions (1)–(4) are satisfied.

(1) The $\mathbf{R}$-linear maps $N_j : H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$ are nilpotent for all j, and $N_j N_k = N_k N_j$ for all j, k.

(2) If $y_j \gg 0$ $(1 \le j \le n)$, then $\exp(\sum_{j=1}^{n} iy_j N_j)F \in D$. (Positivity)

(3) $N_j F^p \subset F^{p-1}$ for all j and p. (Griffiths transversality)

(4) Let J be any subset of $\{1, \ldots, n\}$. Then, for $y_j \in \mathbf{R}_{>0}$ $(j \in J)$, the relative monodromy filtration $M(\sum_{j \in J} y_j N_j, W)$ exists. (Admissibility)

Let $\mathcal{D}_{\mathrm{nilp},n}$ be the set of all $(N_1, \ldots, N_n, F)$ which generate nilpotent orbits.
For $(N_1, \cdots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$, we call the map $(z_1, \ldots, z_n) \mapsto \exp(\sum_{j=1}^{n} z_j N_j)F$ a *nilpotent orbit in n variables*.

In the terminology of Kashiwara [K], $\mathcal{D}_{\mathrm{nilp},n}$ is the set of all $(N_1, \ldots, N_n, F)$ such that $(H_{0,\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \ldots, N_n)$, with $\bar{F}$ the complex conjugate of F, is an "infinitesimal mixed Hodge module".

3.2. We review SL(2)-orbits in the case of pure weight. Let $w \in \mathbf{Z}$ and assume $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$.

Let $n \geq 0$, and consider a pair (ρ, φ) consisting of a homomorphism

$$\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$$

of algebraic groups which is defined over $\mathbf{R}$ and a holomorphic map $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$ satisfying the following condition.

$$\varphi(gz) = \rho(g)\varphi(z) \quad \text{for any } g \in \mathrm{SL}(2, \mathbf{C})^n, \; z \in \mathbf{P}^1(\mathbf{C})^n.$$

As in [KU3], §5 (see also [KU2], §3), we call the pair (ρ, φ) an SL(2)-*orbit in n variables* if it further satisfies the following two conditions (1) and (2).

$$(1) \qquad \varphi(\mathfrak{h}^n) \subset D.$$

$$(2) \qquad \rho_*(F_z^p(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})) \subset F^p_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}}) \quad \text{for any } z \in \mathbf{P}^1(\mathbf{C})^n \text{ and any } p \in \mathbf{Z}.$$

Here in (1), $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$ is an upper-half plane. In (2), ρ_* denotes the Lie algebra homomorphism $\mathfrak{sl}(2, \mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ induced by ρ, and $F_z(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})$, $F_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}})$ are the Hodge filtrations induced by those for $z \in \mathbf{P}^1(\mathbf{C})^n$, $\varphi(z) \in \check{D}$, respectively.

Let

$$Y_j = \rho_* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_j \in \mathfrak{g}_{\mathbf{R}} \quad (1 \leq j \leq n),$$

where $(\;)_j$ means the embedding $\mathfrak{sl}(2) \to \mathfrak{sl}(2)^{\oplus n}$ into the j-th factor.

For $1 \leq j \leq n$, define the increasing filtration $W^{(j)}$ on $H_{0,\mathbf{R}}$ as follows. Let $H_{0,\mathbf{R}} = \bigoplus_{m \in \mathbf{Z}^n} V_m$ be the decomposition, where Y_j acts on V_m as the multiplication by m_j. Let

$$\begin{aligned} W_k^{(j)} &= \textstyle\bigoplus_{m \in \mathbf{Z}^n,\, m_1 + \cdots + m_j \leq k-w} V_m \\ &= (\text{the part of } H_{0,\mathbf{R}} \text{ on which eigen values of } Y_1 + \cdots + Y_j \text{ are} \leq k - w). \end{aligned}$$

Let $s^{(j)}$ be the splitting of $W^{(j)}$ given by the eigen spaces of $Y_1 + \cdots + Y_j$.

Proposition3.3. *An* SL(2)*-orbit in* n *variables is determined by* $((W^{(j)})_{1\le j\le n},\varphi(\mathbf{i}))$.

3.4. In the situation of 3.2, let W' be an increasing filtration on $H_{0,\mathbf{R}}$ such that there exists a group homomorphism $\alpha : \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ such that, for $k \in \mathbf{Z}$, $W'_k = \bigoplus_{m\le k-w} H(m)$, where w is as in 3.2, and $H(m) := \{x \in H_{0,\mathbf{R}} \mid \alpha(t)x = t^m x \quad (t \in \mathbf{R}^\times)\}$.

We define the real analytic map

$$\mathrm{spl}^{\mathrm{BS}}_{W'} : D \to \mathrm{spl}(W')$$

as follows. Let $P = (G^\circ_{W'})_{\mathbf{R}}$ be the parabolic subgroup of $G_{\mathbf{R}}$ defined by W' (G° is the connected component of G as an algebraic group containing 1). Let P_u be the unipotent radical of P, and S_P the maximal $\mathbf{R}$-split torus of the center of P/P_u. Let $\mathbf{G}_{m,\mathbf{R}} \to S_P$, $t \mapsto (t^{k-w}$ on $\mathrm{gr}^{W'}_k)_k$ be the weight map induced by α. For $F \in D$, let K_F be the maximal compact subgroup of $G_{\mathbf{R}}$ consisting of the elements of $G_{\mathbf{R}}$ which preserve the Hodge metric $\langle C_F(\bullet), \bar{\bullet}\rangle_w$, where C_F is the Weil operator associated to F. Let $S_P \to P$ be the Borel-Serre lifting homomorphism at F, which assigns $a \in S_P$ to the element $a_F \in P$ uniquely determined by the following condition: the class of a_F in P/P_u coincides with a, and $\theta_{K_F}(a_F) = a_F^{-1}$, where θ_{K_F} is the Cartan involution at K_F which coincides with $\mathrm{Int}(C_F)$ in the present situation ([KU3] 5.1.3, [KNU1] 8.1). Then, the composite $\mathbf{G}_{m,\mathbf{R}} \to S_P \to P$ defines an action of $\mathbf{G}_{m,\mathbf{R}}$ on $H_{0,\mathbf{R}}$, and we call the corresponding splitting of W' the *Borel-Serre splitting at* F, and denote it by $\mathrm{spl}^{\mathrm{BS}}_{W'}(F)$.

It is easy to see that the map $\mathrm{spl}^{\mathrm{BS}}_{W'} : D \to \mathrm{spl}(W')$, $F \mapsto \mathrm{spl}^{\mathrm{BS}}_{W'}(F)$, is real analytic.

Proposition 3.5. *Let* (ρ,φ) *be an* SL(2)*-orbit in* n *variables, let* $y_j > 0$ $(1 \le j \le n)$, *and let* $p = \varphi(iy_1,\dots,iy_n) \in D$. *Then*

$$s^{(j)} = \mathrm{spl}^{\mathrm{BS}}_{W^{(j)}}(p) \quad (1 \le j \le n).$$

3.6. We now consider SL(2)-orbits in mixed case. Let W be an increasing filtration of $H_{0,\mathbf{R}}$ such that $W_w = 0$ for $w \ll 0$ and $W = H_{0,\mathbf{R}}$ for $w \gg 0$.

For $n \ge 0$, let $\mathcal{D}_{\mathrm{SL}(2),n}$ be the set of all triples $((\rho_w,\varphi_w)_{w\in\mathbf{Z}},\mathbf{r},J)$, where (ρ_w,φ_w) is an SL(2)-orbit in n variables for gr^W_w for each $w \in \mathbf{Z}$, $\mathbf{r}$ is an element of D such that $\mathbf{r}(\mathrm{gr}^W_w) = \varphi_w(\mathbf{i})$ for each $w \in \mathbf{Z}$ ($\mathbf{i} = (i,\dots,i) \in \mathbf{C}^n \subset \mathbf{P}^1(\mathbf{C})^n$), and J is a subset of $\{1,\dots,n\}$ satisfying the following conditions (1) and (2). Let

$$J' = \{j \mid 1 \le j \le n, \text{ there is } w \in \mathbf{Z} \text{ such that the } j\text{-th component } \mathrm{SL}(2) \to G_{\mathbf{R}}(\mathrm{gr}^W_w) \text{ of } \rho_w \text{ is a non-trivial homomorphism}\}.$$

(1) If $\mathbf{r} \in D_{\mathrm{spl}}$, $J = J'$.

(2) If $\mathbf{r} \in D_{\mathrm{nspl}}$, either $J = J'$ or $J = J' \cup \{k\}$ for some $k < \min J'$.

Let

$$\mathcal{D}_{\mathrm{SL}(2)} = \textstyle\bigsqcup_{n\ge0} \mathcal{D}_{\mathrm{SL}(2),n}.$$

We call an element of $\mathcal{D}_{\mathrm{SL}(2),n}$ an SL(2)-*orbit in n variables*, and an element of $\mathcal{D}_{\mathrm{SL}(2)}$ an SL(2)-*orbit*. Note that, in the pure case, J is determined uniquely by $(\rho_w)_w$ since $D = D_{\mathrm{spl}}$.

We call the cardinality of the set J the *rank* of the SL(2)-orbit.

When $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ is an SL(2)-orbit in n variables of rank n, we denote it also simply by $((\rho_w, \varphi_w)_w, \mathbf{r})$ since J is clearly $\{1, \dots, n\}$.

3.7. If $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ is an SL(2)-orbit in n variables of rank r, we have the *associated* SL(2)-*orbit* $((\rho'_w, \varphi'_w)_w, \mathbf{r})$ *in r variables of rank r*, defined as follows. Let $J = \{a(1), \dots, a(r)\}$ with $a(1) < \cdots < a(r)$. Then

$$\rho'_w(g_{a(1)}, \dots, g_{a(r)}) = \rho_w(g_1, \dots, g_n), \quad \varphi'_w(z_{a(1)}, \dots, z_{a(r)}) = \varphi_w(z_1, \dots, z_n).$$

Note that for any $w \in \mathbf{Z}$, ρ_w factors through the projection $\mathrm{SL}(2)^n \to \mathrm{SL}(2)^J$ to the J-component, and φ_w factors through the projection $\mathbf{P}^1(\mathbf{C})^n \to \mathbf{P}^1(\mathbf{C})^J$ to the J-component, and hence $(\rho_w, \varphi_w)_w$ is essentially the same as $(\rho'_w, \varphi'_w)_w$.

3.8. *Associated torus action.*

Assume that we are given an SL(2)-orbit in n variables $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$.

We define the associated homomorphism of algebraic groups over $\mathbf{R}$

$$\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$$

as follows. Let $s_{\mathbf{r}} : \mathrm{gr}^W \xrightarrow{\sim} H_{0,\mathbf{R}}$ be the canonical splitting $\mathrm{spl}_W(\mathbf{r})$ of W associated to $\mathbf{r}$ (§2). Then

$$\tau(t_1, \dots, t_n) = s_{\mathbf{r}} \circ \left(\bigoplus_{w \in \mathbf{Z}} \left(\prod_{j=1}^n t_j\right)^w \rho_w(g_1, \dots, g_n) \text{ on } \mathrm{gr}^W_w\right) \circ s_{\mathbf{r}}^{-1}$$

$$\text{with} \quad g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix}.$$

For $1 \le j \le n$, let $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the j-th component of τ.

Remark. The induced action of $\tau(t)$ ($t \in \mathbf{R}^n_{>0}$) on D is described as follows. For $s(\theta(F, \delta)) \in D$ with $s \in \mathrm{spl}(W)$, $F \in D(\mathrm{gr}^W)$, $\delta \in \mathcal{L}(F)$ (§2), we have

$$\tau(t)s(\theta(F, \delta)) = s'(\theta(F', \delta'))$$

with $s' = \tau(t)s\,\mathrm{gr}^W(\tau(t))^{-1}$, $F' = \mathrm{gr}^W(\tau(t))F$, $\delta' = \mathrm{Ad}(\mathrm{gr}^W(\tau(t)))\delta$.

3.9. *Associated family of weight filtrations.*

In the situation of 3.8, for $1 \le j \le n$, we define the associated j-th weight filtration $W^{(j)}$ on $H_{0,\mathbf{R}}$ as follows. For $k \in \mathbf{Z}$, $W^{(j)}_k$ is the direct sum of $\{x \in H_{0,\mathbf{R}} \mid \tau_j(t)x = t^\ell x \ (\forall\, t \in \mathbf{R}^\times)\}$ over all $\ell \le k$.

By definition, we have $W^{(j)}_k = \sum_{w \in \mathbf{Z}} s_{\mathbf{r}}(W^{(j)}_k(\mathrm{gr}^W_w))$, and $W^{(j)}_k(\mathrm{gr}^W_w)$ coincides with the k-th filter of the j-th weight filtration on gr^W_w associated to the SL(2)-orbit (ρ_w, φ_w) in n variables.

Proposition 3.10. (i) *An* SL(2)*-orbit in* n *variables* $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ *is uniquely determined by* $((W^{(j)}(\mathrm{gr}^W))_{1\le j\le n}, \mathbf{r}, J)$.

(ii) *An* SL(2)*-orbit in* n *variables* $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ *is uniquely determined by* $(\tau, \mathbf{r}, J)$.

3.11. For an increasing filtration W' on a finite dimensional vector space E such that $W'_w = E$ for $w \gg 0$ and $W'_w = 0$ for $w \ll 0$, define the *mean value of the weights* $\mu(W') \in \mathbf{Q}$ *of* W' and the *variance of the weights* $\sigma^2(W') \in \mathbf{Q}$ *of* W' by

$$\mu(W') = \textstyle\sum_{w\in\mathbf{Z}} \dim(\mathrm{gr}_w^{W'})w/\dim(E),$$
$$\sigma^2(W') = \textstyle\sum_{w\in\mathbf{Z}} \dim(\mathrm{gr}_w^{W'})(w-\mu(W'))^2/\dim(E).$$

Proposition 3.12. *Let* $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ *be an* SL(2)*-orbit in* n *variables, and let* $W^{(j)}$ $(1 \le j \le n)$ *be as in* 3.9. *Let* $W^{(0)} = W$.

(i) *Let* $1 \le j \le n$. *Then* $W^{(j)} = W^{(j-1)}$ *if and only if for any* $w \in \mathbf{Z}$, *the* j*-th factor* $\mathrm{SL}(2,\mathbf{C}) \to G_{\mathbf{C}}(\mathrm{gr}_w^W)$ *of* ρ_w *is the trivial homomorphism.*

(ii) *For* $0 \le j \le n$, *let* $\sigma^2(j) = \sum_{w\in\mathbf{Z}} \sigma^2(W^{(j)}(\mathrm{gr}_w^W))$. *Then,* $\sigma^2(j) \le \sigma^2(j')$ *if* $0 \le j \le j' \le n$.

(iii) *Let* $0 \le j \le n$, $0 \le j' \le n$. *Then,* $W^{(j)} = W^{(j')}$ *if and only if* $\sigma^2(j) = \sigma^2(j')$.

§4. SL(2)-ORBIT THEOREM IN MIXED CASE

SL(2)-orbit theorem in [KNU1] is rewritten as

Theorem 4.1. *There is a map*

$$\psi : \mathcal{D}_{\mathrm{nilp},n} \to \mathcal{D}_{\mathrm{SL}(2),n}, \quad (N_1, \ldots, N_n, F) \mapsto ((\rho_w, \varphi_w)_w, \mathbf{r}_1, J),$$

defined as follows.

Let $(N_1, \ldots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$. *For each* $w \in \mathbf{Z}$, *let* (ρ_w, φ_w) *be the* SL(2)*-orbit in* n *variables for* gr_w^W *associated to* $(\mathrm{gr}_w^W(N_1), \ldots, \mathrm{gr}_w^W(N_n), F(\mathrm{gr}_w^W))$, *which generates a nilpotent orbit for* gr_w^W. *Let* $k = \min(\{j \mid 1 \le j \le n,\ N_j \ne 0\} \cup \{n+1\})$.

(i) *If* $y_j \in \mathbf{R}_{>0}$ *and* $y_j/y_{j+1} \to \infty$ $(1 \le j \le n$, y_{n+1} *means* $1)$, *the canonical splitting* $\mathrm{spl}_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ *of* W *associated to* $\exp(\sum_{j=1}^n iy_jN_j)F$ (§2) *converges in* $\mathrm{spl}(W)$.

Let $s \in \mathrm{spl}(W)$ *be the limit.*

(ii) *Let* $\tau : \mathbf{G}_{m,\mathbf{R}}^n \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ *be the homomorphism of algebraic groups defined by*

$$\tau(t_1, \ldots, t_n) = s \circ \Big(\bigoplus_{w\in\mathbf{Z}} \Big(\big(\textstyle\prod_{j=1}^n t_j\big)^w \rho_w(g_1, \ldots, g_n) \text{ on } \mathrm{gr}_w^W\Big)\Big) \circ s^{-1},$$

where g_j is as in 3.8. *Then, as $y_j > 0$, $y_1 = \cdots = y_k$, $y_j/y_{j+1} \to \infty$ ($k \le j \le n$, y_{n+1} means* 1),

$$\tau\Big(\sqrt{\frac{y_2}{y_1}}, \ldots, \sqrt{\frac{y_{n+1}}{y_n}}\Big)^{-1} \exp(\textstyle\sum_{j=1}^n iy_jN_j)F$$

converges in D.

Let $\mathbf{r}_1 \in D$ be the limit.

(iii) *Let*

$$J' = \{j \mid 1 \le j \le n, \text{ the } j\text{-th component of } \rho_w \text{ is non-trivial for some } w \in \mathbf{Z}\}.$$

For k defined at the beginning, let $J = J' = \varnothing$ if $k = n+1$, and let $J = J' \cup \{k\}$ if otherwise. Then

$$((\rho_w, \varphi_w)_w, \mathbf{r}_1, J) \in \mathcal{D}_{\mathrm{SL}(2),n}.$$

For $p \in \mathcal{D}_{\mathrm{nilp},n}$, we call $\psi(p) \in \mathcal{D}_{\mathrm{SL}(2),n}$ the SL(2)*-orbit associated to p*.

4.2. For $p = (N_1, \ldots, N_n, F) \in \mathcal{D}_{\mathrm{nilp},n}$, τ in 4.1 (ii) and $(M(N_1 + \cdots + N_j, W))_{1 \le j \le n}$ coincide with the torus action (3.8) and the family of weight filtrations (3.9) associated to $\psi(p) \in \mathcal{D}_{\mathrm{SL}(2),n}$, respectively.

The map ψ in 4.1 is not necessarily surjective, e.g. in Example III in 9.1 below, $\mathrm{Im}\,\psi$ is contained in Case 1, and the SL(2)-orbits in Case 2 and in Case 3 are away from $\mathrm{Im}\,\psi$.

Note that, in the definition of a nilpotent orbit in 3.1, the order of $N_1, \ldots, N_n$ in $(N_1, \cdots, N_n, F)$ is not important, but when we consider the SL(2)-orbit associated to $(N_1, \cdots, N_n, F)$, the order of $N_1, \ldots, N_n$ becomes essential.

Note that, even when $k = 1$, $\mathbf{r}_1$ in 4.1 (ii) above is not $\mathbf{r}$ but $\exp(\varepsilon_0)\mathbf{r}$ in Theorem 0.5 of [KNU1], and that s in 4.1 (i) above coincides with $\mathrm{spl}_W(\mathbf{r}_1)$ (§2).

§5. Set $D_{\mathrm{SL}(2)}$

5.1. Two SL(2)-orbits $p = ((\rho_w, \varphi_w)_w, \mathbf{r}), p' = ((\rho'_w, \varphi'_w)_w, \mathbf{r}') \in \mathcal{D}'_{\mathrm{SL}(2),n}$ in n variables of rank n (3.6) are said to be *equivalent* if there is a $t \in \mathbf{R}^n_{>0}$ such that

$$\rho'_w = \mathrm{Int}(\mathrm{gr}^W_w(\tau(t))) \circ \rho_w, \quad \varphi'_w = \mathrm{gr}^W_w(\tau(t)) \circ \varphi_w \quad (\forall\, w \in \mathbf{Z}), \quad \mathbf{r}' = \tau(t)\mathbf{r}.$$

Here $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the torus action associated to $((\rho_w, \varphi_w)_w, \mathbf{r})$ (3.8).

Two SL(2)-orbits $((\rho_w, \varphi_w)_w, \mathbf{r}, J), ((\rho'_w, \varphi'_w)_w, \mathbf{r}', J') \in \mathcal{D}_{\mathrm{SL}(2)}$ in n variables of rank r, and in n' variables of rank r', respectively, are said to be *equivalent* if $r = r'$ and their associated SL(2)-orbits in r variables of rank r (3.7) are equivalent.

Proposition 5.2. *Let $p = ((\rho_w, \varphi_w)_w, \mathbf{r})$ be an* SL(2)*-orbit in n variables of rank n.*

(i) *The $W^{(j)}$ of p, the τ and the τ_j of p ($1 \le j \le n$), the canonical splitting* $\mathrm{spl}_W(\mathbf{r})$ *of W associated to* $\mathbf{r}$ (§2), *and $Z = \tau(\mathbf{R}^n_{>0})\mathbf{r}$ depend only on the equivalence class of p. Z is called the* torus orbit associated to p.

(ii) *The equivalence class of p is determined by $((W^{(j)}(\mathrm{gr}^W))_{1\le j\le n}, Z)$, where Z is as above.*

(iii) *The equivalence class of p is determined by (τ, Z), where τ and Z are as above.*

5.3. Let $D_{\mathrm{SL}(2)}$ be the set of all equivalence classes of SL(2)-orbits satisfying the following condition (1).

Take an SL(2)-orbit $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ in n variables which is a representative of the class in question.

(1) For each $w \in \mathbf{Z}$ and for each $1 \le j \le n$, the weight filtration $W^{(j)}(\mathrm{gr}_w^W)$ is rational.

As a set, we have

$$D_{\mathrm{SL}(2)} = \bigsqcup_{n\ge 0} D_{\mathrm{SL}(2),n},$$

where $D_{\mathrm{SL}(2),n}$ is the set of equivalence classes of SL(2)-orbits of rank n with rational associated weight filtrations. We identify $D_{\mathrm{SL}(2),0}$ with D in the evident way.

Let $D_{\mathrm{SL}(2),\mathrm{spl}}$ be the subset of $D_{\mathrm{SL}(2)}$ consisting of the classes of $((\rho_w, \varphi_w)_w, \mathbf{r}, J)$ with $\mathbf{r} \in D_{\mathrm{spl}}$. Let $D_{\mathrm{SL}(2),\mathrm{nspl}} = D_{\mathrm{SL}(2)} \smallsetminus D_{\mathrm{SL}(2),\mathrm{spl}}$.

5.4. We have a canonical projection

$$D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = \prod_{w\in\mathbf{Z}} D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W),$$

$$\mathrm{class}((\rho_w, \varphi_w)_w, \mathbf{r}, J) \mapsto (\mathrm{class}(\rho_w, \varphi_w))_w.$$

Here $D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)$ is the $D_{\mathrm{SL}(2)}$ for $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w)$. Note that, in the pure case, the definition of $D_{\mathrm{SL}(2)}$ coincides with that of [KU2].

We have a canonical map

$$D_{\mathrm{SL}(2)} \to \mathrm{spl}(W), \quad \mathrm{class}((\rho_w, \varphi_w)_w, \mathbf{r}, J) \mapsto s,$$

where s denotes the canonical splitting of W associated to $\mathbf{r}$ (see 5.2 (i)).

5.5. For $p \in D_{\mathrm{SL}(2)}$, we denote by τ_p and Z_p the corresponding τ and Z, respectively.

Later in §7, we will define two topologies on the set $D_{\mathrm{SL}(2)}$. Basic properties of these topologies are the following.

(i) If $p \in D_{\mathrm{SL}(2)}$ is the class of $(\tau_p, \mathbf{r})$ (cf. 5.2 (iii)), then we have in $D_{\mathrm{SL}(2)}$

$$\tau_p(t)\mathbf{r} \to p \quad \text{when } t \in \mathbf{R}^n_{>0} \text{ tends to } \mathbf{0}.$$

Here n is the rank of p and $\mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^n_{\ge 0}$.

(ii) If $(N_1, \cdots, N_n, F)$ generates a nilpotent orbit and if the monodromy filtration of $\mathrm{gr}_w^W(N_1) + \cdots + \mathrm{gr}_w^W(N_j)$ is rational for any $w \in \mathbf{Z}$ and any $1 \le j \le n$, then we have in $D_{\mathrm{SL}(2)}$

$$\exp(\textstyle\sum_{j=1}^n iy_jN_j)F \to p$$

when $y_j > 0$, $y_j/y_{j+1} \to \infty$ ($1 \le j \le n$, y_{n+1} denotes 1), where p denotes the class of the SL(2)-orbit associated to $(N_1, \dots, N_n, F)$ by 4.2.

This (ii) is the basic principle which lies in our construction of the topologies on $D_{\mathrm{SL}(2)}$. Our SL(2)-orbit theorem 4.1 says roughly that, when $y_j/y_{j+1} \to \infty$ ($1 \le j \le n$, $y_{n+1} = 1$), $\exp(\sum_{j=1}^{n} iy_jN_j)F$ is near to $\tau_p(\sqrt{\frac{y_2}{y_1}}, \dots, \sqrt{\frac{y_{n+1}}{y_n}})\mathbf{r}$, where $\mathbf{r} \in Z_p$. Hence (i) is natural in view of (ii).

§6. Category $\mathcal{B}_{\mathbf{R}}(\log)$

6.1. We define a *category $\mathcal{B}_{\mathbf{R}}$ of spaces with real analytic structures.* An object of $\mathcal{B}_{\mathbf{R}}$ is a local ringed space $(S, \mathcal{O}_S)$ over $\mathbf{R}$ such that the following holds locally on S. There are $n \ge 0$ and a morphism $\iota : S \to \mathbf{R}^n$ of local ringed spaces over $\mathbf{R}$ from S to the real analytic manifold $\mathbf{R}^n$ such that ι is injective, the topology of S coincides with the one induced from the topology of $\mathbf{R}^n$ via ι, and the canonical map $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_S$ is surjective and the kernel is a finitely generated ideal. Here $\mathcal{O}_{\mathbf{R}^n}$ denotes the sheaf of $\mathbf{R}$-valued real analytic functions on $\mathbf{R}^n$ and $\iota^{-1}(\ )$ denotes the inverse image of a sheaf. Morphisms of $\mathcal{B}_{\mathbf{R}}$ are those of local ringed spaces over $\mathbf{R}$.

Of course a real analytic manifold is an object of $\mathcal{B}_{\mathbf{R}}$. An example of an object of $\mathcal{B}_{\mathbf{R}}$ which often appears in this note is $\mathbf{R}^n_{\ge 0}$ with the inverse image of the sheaf of real analytic functions on $\mathbf{R}^n$.

For an object $(S, \mathcal{O}_S)$ of $\mathcal{B}_{\mathbf{R}}$, we often call $\mathcal{O}_S$ the sheaf of real analytic functions of S though $(S, \mathcal{O}_S)$ need not be a real analytic space.

6.2. For $(S, \mathcal{O}_S) \in \mathcal{B}_{\mathbf{R}}$, let $\mathcal{O}^\times_{S,>0}$ be the subsheaf of $\mathcal{O}^\times_S$ consisting of all local sections whose values are > 0. A *log structure with sign* on S is an integral log structure M_S on S in the sense of Fontaine-Illusie ([KU3], §2.1) endowed with a subgroup sheaf $M^{\mathrm{gp}}_{S,>0}$ of M^{gp}_S satisfying the following three conditions (1)–(3). Here $M^{\mathrm{gp}}_S \supset M_S$ denotes the sheaf of commutative groups $\{ab^{-1} \mid a, b \in M_S\}$ associated to the sheaf M_S of commutative monoids.

(1) $M^{\mathrm{gp}}_{S,>0} \supset \mathcal{O}^\times_{S,>0}$.

(2) $\mathcal{O}^\times_S/\mathcal{O}^\times_{S,>0} \xrightarrow{\sim} M^{\mathrm{gp}}_S/M^{\mathrm{gp}}_{S,>0}$.

(3) Let $M_{S,>0} := M_S \cap M^{\mathrm{gp}}_{S,>0} \subset M^{\mathrm{gp}}_S$. Then the image of $M_{S,>0}$ in $\mathcal{O}_S$ under the structural map $M_S \to \mathcal{O}_S$ of the log structure has values in $\mathbf{R}_{\ge 0} \subset \mathbf{R}$ at any points of S. (We remark $(M_{S,>0})^{\mathrm{gp}} = M^{\mathrm{gp}}_{S,>0}$ and thus $M^{\mathrm{gp}}_{S,>0}$ is recovered from $M_{S,>0}$.)

Define $\mathcal{B}_{\mathbf{R}}(\log)$ to be the category of objects of $\mathcal{B}_{\mathbf{R}}$ endowed with an fs log structure ([KU3], §2.1) with sign.

6.3. As standard examples of objects of $\mathcal{B}_{\mathbf{R}}(\log)$, we have real toric varieties, and also real analytic manifolds with corners.

Let P be an fs monoid. Then we have

$$|\mathrm{toric}|_P := \mathrm{Hom}(P, \mathbf{R}^{\mathrm{mult}}_{\ge 0}) \subset \mathrm{toric}_P := \mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}} = \mathrm{Hom}(P, \mathbf{C}^{\mathrm{mult}}),$$

where the log structure M of $|\mathrm{toric}|_P$ is the one associated to $P \to \mathcal{O}$. $M_{>0}^{\mathrm{gp}}$ is generated by P^{gp} and $\mathcal{O}_{>0}^{\times}$.

In the case $P = \mathbf{N}^n$, we have $|\mathrm{toric}|_P = \mathbf{R}_{\geq 0}^n$.

A real analytic manifold with corners S is a local ringed space over $\mathbf{R}$ which has an open covering $(U_\lambda)_\lambda$ such that for each λ, U_λ is isomorphic to an open set of the object $\mathbf{R}_{\geq 0}^{n(\lambda)}$ of $\mathcal{B}_{\mathbf{R}}(\log)$ for some $n(\lambda) \geq 0$. The inverse images on U_λ of the fs log structures with sign of $\mathbf{R}_{\geq 0}^{n(\lambda)}$ glue together to an fs log structure with sign on S canonically. Thus a real analytic manifold with corners is regarded canonically as an object of $\mathcal{B}_{\mathbf{R}}(\log)$.

§7. Real analytic structures of $D_{\mathrm{SL}(2)}$

In this section, we define on the set $D_{\mathrm{SL}(2)}$ two structures as object of $\mathcal{B}_{\mathbf{R}}(\log)$.

7.1. We define sets $\mathcal{W}$, $\overline{\mathcal{W}}$, a subset $D_{\mathrm{SL}(2)}^{I}(\Psi)$ of $D_{\mathrm{SL}(2)}$ for $\Psi \in \mathcal{W}$, and a subset $D_{\mathrm{SL}(2)}^{II}(\Phi)$ of $D_{\mathrm{SL}(2)}$ for $\Phi \in \overline{\mathcal{W}}$, as follows.

For $p \in D_{\mathrm{SL}(2)}$, let $\mathcal{W}(p)$ be the set of weight filtrations associated to p.

By an *admissible set of weight filtrations on* $H_{0,\mathbf{R}}$, we mean a finite set Ψ of increasing filtrations on $H_{0,\mathbf{R}}$ such that $\Psi = \mathcal{W}(p)$ for some element p of $D_{\mathrm{SL}(2)}$. We denote by $\mathcal{W}$ the set of all admissible sets of weight filtrations on $H_{0,\mathbf{R}}$.

For $\Psi \in \mathcal{W}$, we define a subset $D_{\mathrm{SL}(2)}^{I}(\Psi)$ of $D_{\mathrm{SL}(2)}$ by

$$D_{\mathrm{SL}(2)}^{I}(\Psi) = \{p \in D_{\mathrm{SL}(2)} \mid \mathcal{W}(p) \subset \Psi\}.$$

Note that $D_{\mathrm{SL}(2)}$ is covered by the subsets $D_{\mathrm{SL}(2)}^{I}(\Psi)$ for $\Psi \in \mathcal{W}$. Furthermore, $D_{\mathrm{SL}(2)}$ is covered by the subsets $D_{\mathrm{SL}(2)}^{I}(\Psi)$ for $\Psi \in \mathcal{W}$ with $W \notin \Psi$ and the subsets $D_{\mathrm{SL}(2)}^{I}(\Psi)_{\mathrm{nspl}} := D_{\mathrm{SL}(2)}^{I}(\Psi) \cap D_{\mathrm{SL}(2),\mathrm{nspl}}$ for $\Psi \in \mathcal{W}$ with $W \in \Psi$. As is stated in Theorem 7.7 below, these are open coverings of $D_{\mathrm{SL}(2)}$ for the topology of $D_{\mathrm{SL}(2)}^{I}$.

For $p \in D_{\mathrm{SL}(2)}$, let

$$\overline{\mathcal{W}}(p) = \{W'(\mathrm{gr}^W) \mid W' \in \mathcal{W}(p),\ W' \neq W\},$$

where $W'(\mathrm{gr}^W)$ is the filtration on $\mathrm{gr}^W = \bigoplus_w \mathrm{gr}_w^W$ induced by W', i.e., $W'(\mathrm{gr}^W)_k := \bigoplus_w W'_k(\mathrm{gr}_w^W) \subset \bigoplus_w \mathrm{gr}_w^W$.

By an *admissible set of weight filtrations on* gr^W, we mean a finite set Φ of increasing filtrations on gr^W such that $\Phi = \overline{\mathcal{W}}(p)$ for some element p of $D_{\mathrm{SL}(2)}$. We denote by $\overline{\mathcal{W}}$ the set of all admissible sets of weight filtrations on gr^W.

For $\Phi \in \overline{\mathcal{W}}$, we define a subset $D_{\mathrm{SL}(2)}^{II}(\Phi)$ of $D_{\mathrm{SL}(2)}$ by

$$D_{\mathrm{SL}(2)}^{II}(\Phi) = \{p \in D_{\mathrm{SL}(2)} \mid \overline{\mathcal{W}}(p) \subset \Phi\}.$$

As a set, $D_{\mathrm{SL}(2)}$ is covered by $D_{\mathrm{SL}(2)}^{II}(\Phi)$ $(\Phi \in \overline{\mathcal{W}})$. As is stated in Theorem 7.7 below, this is an open covering for the topology of $D_{\mathrm{SL}(2)}^{II}$.

We have a canonical map
$$\mathcal{W} \to \overline{\mathcal{W}}$$
which sends $\Psi \in \mathcal{W}$ to $\bar{\Psi} := \{W'(\mathrm{gr}^W) \mid W' \in \Psi,\ W' \neq W\} \in \overline{\mathcal{W}}$. For $\Psi \in \mathcal{W}$, we have $D^I_{\mathrm{SL}(2)}(\Psi) \subset D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$.

7.2. Let $\Psi \in \mathcal{W}$. A homomorphism $\alpha : \mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ of algebraic groups over $\mathbf{R}$ is called a *splitting of* Ψ if it satisfies the following conditions (1)–(2).

(1) The corresponding direct sum decomposition $H_{0,\mathbf{R}} = \bigoplus_{\mu} S_{\mu}$ into eigen $\mathbf{R}$-subspaces S_{μ} ($\mu \in \mathbf{Z}^{\Psi}$) satisfies $W'_{w'} = \sum_{\mu(W') \leq w'} S_{\mu}$ for all $W' \in \Psi$ and all $w' \in \mathbf{Z}$.

(2) For all $w \in \mathbf{Z}$, the image of $\iota^{-w}\alpha'_w$ is contained in $G_{\mathbf{R}}(\mathrm{gr}^W_w)$, where $\alpha'_w : \mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ is the homomorphism induced by α, and ι is the composite of the multiplication $\mathbf{G}^{\Psi}_{m,\mathbf{R}} \to \mathbf{G}_{m,\mathbf{R}}$ and the canonical map $\mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$.

A splitting of Ψ exists: If Ψ is associated to $p \in D_{\mathrm{SL}(2)}$, the torus action τ_p associated to p is a splitting of Ψ. Here and hereafter, we identify $\{1, \dots, n\}$ (n is the rank of p) with Ψ via the bijection $j \mapsto W^{(j)}$, which is independent of the choice of p by 3.12.

Let $\Phi \in \overline{\mathcal{W}}$. A homomorphism $\alpha : \mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ of algebraic groups over $\mathbf{R}$ is called a *splitting of* Φ if it satisfies the following conditions ($\bar{1}$)–($\bar{2}$).

($\bar{1}$) The corresponding direct sum decomposition $\mathrm{gr}^W = \bigoplus_{\mu} S_{\mu}$ into eigen $\mathbf{R}$-subspaces S_{μ} ($\mu \in \mathbf{Z}^{\Phi}$) satisfies $W'_{w'} = \sum_{\mu(W') \leq w'} S_{\mu}$ for all $W' \in \Phi$ and all $w' \in \mathbf{Z}$.

($\bar{2}$) For all $w \in \mathbf{Z}$, the image of $\iota^{-w}\alpha'_w$ is contained in $G_{\mathbf{R}}(\mathrm{gr}^W_w)$, where $\alpha'_w : \mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ is the w-component of α, and ι is the composite of the multiplication $\mathbf{G}^{\Phi}_{m,\mathbf{R}} \to \mathbf{G}_{m,\mathbf{R}}$ and the canonical map $\mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$.

A splitting of Φ exists: For $p \in D_{\mathrm{SL}(2)}$, let $\bar{\tau}_p$ be $\mathrm{gr}^W(\tau_p)$ in the case $W \notin \mathcal{W}(p)$, and in the case $W \in \mathcal{W}(p)$, let $\bar{\tau}_p$ be the restriction of $\mathrm{gr}^W(\tau_p)$ to $\mathbf{G}^{\overline{\mathcal{W}(p)}}_{m,\mathbf{R}}$ which we identify with the part of $\mathbf{G}^{\mathcal{W}(p)}_{m,\mathbf{R}}$ with the W-component removed. Then if $\Phi = \overline{\mathcal{W}}(p)$, $\bar{\tau}_p$ is a splitting of Φ.

7.3. Let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp. $W \in \Psi$). If a real analytic map $\beta : D \to \mathbf{R}^{\Psi}_{>0}$ (*resp.* $D_{\mathrm{nspl}} \to \mathbf{R}^{\Psi}_{>0}$) satisfies the following (1) for any splitting α of Ψ, then we call β a *distance to* Ψ*-boundary.*

$$(1) \qquad \beta(\alpha(t)p) = t\beta(p) \quad (t \in \mathbf{R}^{\Psi}_{>0},\ p \in D \text{ (resp. } D_{\mathrm{nspl}})).$$

A distance to Ψ-boundary exists.

Let $\Phi \in \overline{\mathcal{W}}$. If a real analytic map $\beta : D(\mathrm{gr}^W) \to \mathbf{R}^{\Phi}_{>0}$ satisfies the following ($\bar{1}$) for any splitting α of Φ, then we call β a *distance to* Φ*-boundary.*

$$(\bar{1}) \qquad \beta(\alpha(t)p) = t\beta(p) \quad (t \in \mathbf{R}^{\Phi}_{>0},\ p \in D(\mathrm{gr}^W)).$$

A distance to Φ-boundary exists.

Theorem 7.4. (i) *Let $\Psi \in \mathcal{W}$, let α be a splitting of Ψ, and let β be a distance to Ψ-boundary. Assume $W \notin \Psi$ (resp. $W \in \Psi$) and consider the map*

$$\nu_{\alpha,\beta} : D \ (\textit{resp. } D_{\mathrm{nspl}}) \to \mathbf{R}^{\Psi}_{>0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}(W'(\mathrm{gr}^W)),$$

$$p \mapsto (\beta(p),\ \alpha\beta(p)^{-1}p,\ \mathrm{spl}_W(p),\ (\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W)))_{W' \in \Psi}).$$

Here $\mathrm{spl}_W(p)$ is the canonical splitting of W associated to p (§2) and $\mathrm{spl}^{\mathrm{BS}}_{W'(\mathrm{gr}^W)}(p(\mathrm{gr}^W))$ is the Borel-Serre splitting of $W'(\mathrm{gr}^W)$ associated to $p(\mathrm{gr}^W)$ (3.4). Let $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$), J the set of weight filtrations associated to p (3.9), $\tau_p : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ the associated torus action (3.8), and $\mathbf{r} \in D$ a point on the torus orbit (5.2) associated to p. Then, when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$, $\nu_{\alpha,\beta}(\tau_p(t)\mathbf{r})$ converges in $B := \mathbf{R}^{\Psi}_{\geq 0} \times D \times \mathrm{spl}(W) \times \prod_{W' \in \Psi} \mathrm{spl}(W'(\mathrm{gr}^W))$. This limit depends only on p and is independent of the choice of $\mathbf{r}$.

The extended map $\nu_{\alpha,\beta} : D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) $\to B$ is injective.

(ii) *Let $\Phi \in \overline{\mathcal{W}}$, let α be a splitting of Φ, and let β be a distance to Φ-boundary. Consider the map*

$$\nu_{\alpha,\beta} : D \to \mathbf{R}^{\Phi}_{>0} \times D(\mathrm{gr}^W) \times \mathcal{L} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W'),$$

$$p \mapsto (\beta(p(\mathrm{gr}^W)),\ \alpha\beta(p(\mathrm{gr}^W))^{-1}p(\mathrm{gr}^W),\ \mathrm{Ad}(\alpha\beta(p(\mathrm{gr}^W)))^{-1}\delta(p),$$
$$\mathrm{spl}_W(p),\ (\mathrm{spl}^{\mathrm{BS}}_{W'}(p(\mathrm{gr}^W)))_{W' \in \Phi}).$$

Here $\mathcal{L}$ is in §3 and $\delta(p)$ denotes δ of p. Let $p \in D^{II}_{\mathrm{SL}(2)}(\Phi)$, J the set of weight filtrations associated to p, $\tau_p : \mathbf{G}^J_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ the associated torus action, and $\mathbf{r} \in D$ a point on the torus orbit associated to p. Then, when $t \in \mathbf{R}^J_{>0}$ tends to 0^J in $\mathbf{R}^J_{\geq 0}$, $\nu_{\alpha,\beta}(\tau_p(t)\mathbf{r})$ converges in $B := \mathbf{R}^{\Phi}_{\geq 0} \times D(\mathrm{gr}^W) \times \bar{\mathcal{L}} \times \mathrm{spl}(W) \times \prod_{W' \in \Phi} \mathrm{spl}(W')$. This limit depends only on p and is independent of the choice of $\mathbf{r}$.

The extended map $\nu_{\alpha,\beta} : D^{II}_{\mathrm{SL}(2)}(\Phi) \to B$ is injective.

7.5. On $\bar{\mathcal{L}}$ in 7.4 (ii).

We recall the compactified vector space $\bar{V}$ associated to a weightened finite dimensional $\mathbf{R}$-vector space $V = \bigoplus_{w \in \mathbf{Z}} V_w$ such that $V_w = 0$ unless $w \leq -1$. It is a compact real analytic manifold with boundary. For $t \in \mathbf{R}_{>0}$ and $v = \sum_{w \in \mathbf{Z}} v_w \neq 0$ ($v_w \in V_w$), let $t \circ v = \sum_w t^w v_w$. Then as a set, $\bar{V}$ is the disjoint union of V and the points $0 \circ v$ ($v \in V \smallsetminus \{0\}$), where $0 \circ v$ is the limit point in $\bar{V}$ of $t \circ v$ with $t \in \mathbf{R}_{>0}$, $t \to 0$. $0 \circ v = 0 \circ v'$ if and only if $v' = t \circ v$ for some $t \in \mathbf{R}_{>0}$.

Since $\bar{V}$ is a real analytic manifold with boundary (a special case of a real analytic manifold with corners), $\bar{V}$ is regarded as an object of $\mathcal{B}(\log)$.

Since $\mathcal{L}$ is a finite dimensional weightened $\mathbf{R}$-vector space of weights ≤ -2, we have the associated compactified vector space $\bar{\mathcal{L}} \supset \mathcal{L}$.

In 7.4 (ii), the $\bar{\mathcal{L}}$-component of $\nu_{\alpha,\beta}(p)$ belongs to $\mathcal{L}$ (resp. $\bar{\mathcal{L}} \setminus \mathcal{L}$) if and only if $W \notin \mathcal{W}(p)$ (resp. $W \in \mathcal{W}(p)$).

7.6. Let $\Psi \in \mathcal{W}$. Assume $W \notin \Psi$ (resp. $W \in \Psi$). Let $A = D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $A = D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$), and let B be as in 7.4 (i). Regard B as an object of $\mathcal{B}_{\mathbf{R}}(\log)$. Define the topology of A to be the one as a subspace of B in which A is embedded by $\nu_{\alpha,\beta}$ in 7.4 (i). We define the sheaf of real analytic functions on A as follows. For an open set U of A and a function $f : U \to \mathbf{R}$, we say f is a real analytic function if and only if for each $p \in U$, there are an open neighborhood U' of p in U, an open neighborhood U'' of U' in B, and a real analytic function $g : U'' \to \mathbf{R}$ such that the restrictions to U' of f and g coincide. Define the log structure with sign on A as the inverse image of the log structure with sign of B.

Let $\Phi \in \overline{\mathcal{W}}$. Let $A = D^{II}_{\mathrm{SL}(2)}(\Phi)$, B be as in 7.4 (ii). Regard B as an object of $\mathcal{B}_{\mathbf{R}}(\log)$. Define the topology of A to be the one as a subspace of B in which A is embedded by $\nu_{\alpha,\beta}$ in 7.4 (ii). We define the sheaf of real analytic functions on A as follows. For an open set U of A and a function $f : U \to \mathbf{R}$, we say f is a real analytic function if and only if, for each $p \in U$, there are an open neighborhood U' of p in U, an open neighborhood U'' of U' in B, and a real analytic function $g : U'' \to \mathbf{R}$ such that the restrictions to U' of f and g coincide. Define the log structure with sign on A as the inverse image of the log structure with sign of B.

Theorem 7.7. (i) *There exists a unique structure $D^I_{\mathrm{SL}(2)}$ of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ on the set $D_{\mathrm{SL}(2)}$ having the following property: For any $\Psi \in \mathcal{W}$, $D^I_{\mathrm{SL}(2)}(\Psi)$ and $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$ are open in $D^I_{\mathrm{SL}(2)}$, and if $W \notin \Psi$ (resp. $W \in \Psi$), the induced structure on $D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) of local ringed space over $\mathbf{R}$ endowed with log structure with sign coincides with the one in* 7.6.

(ii) *There exists a unique structure $D^{II}_{\mathrm{SL}(2)}$ of an object of $\mathcal{B}_{\mathbf{R}}(\log)$ on the set $D_{\mathrm{SL}(2)}$ having the following property: For any $\Phi \in \overline{\mathcal{W}}$, $D^{II}_{\mathrm{SL}(2)}(\Phi)$ is open in $D^{II}_{\mathrm{SL}(2)}$, and the induced structure on $D_{\mathrm{SL}(2)}(\Phi)$ of local ringed space over $\mathbf{R}$ endowed with log structure with sign coincides with the one in* 7.6.

(iii) *We have a morphism $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ in $\mathcal{B}_{\mathbf{R}}(\log)$ whose underlying map of sets is the identity map of $D_{\mathrm{SL}(2)}$. This morphism is strict, i.e., the log structure with sign on $D^I_{\mathrm{SL}(2)}$ coincides with the inverse image of that of $D^{II}_{\mathrm{SL}(2)}$.*

In the pure case, this morphism is an isomorphism, and the topology of $D_{\mathrm{SL}(2)}$ given by these structures coincides with the one defined in [KU2].

Proposition 7.8. *The following conditions* (1)–(3) *are equivalent.*

(1) *The topology of $D^I_{\mathrm{SL}(2)}$ coincides with that of $D^{II}_{\mathrm{SL}(2)}$.*

(2) *$D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ coincide in $\mathcal{B}_{\mathbf{R}}(\log)$.*

(3) *For any $p \in D_{\mathrm{SL}(2)}$, for any $w, w' \in \mathbf{Z}$ such that $w > w'$, for any member W' of the set of weight filtrations associated to p, and for any $a, b \in \mathbf{Z}$ such that $\mathrm{gr}^{W'}_a(\mathrm{gr}^W_w) \neq 0$ and $\mathrm{gr}^{W'}_b(\mathrm{gr}^W_{w'}) \neq 0$, we have $a \geq b$.*

Theorem 7.9. *The canonical map*

$$D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \times D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$$

is proper.

Theorem 7.10. *Let Γ be a subgroup of $G_{\mathbf{Z}}$. For $* = I, II$, we have the following.*

(i) *The action of Γ on $D^*_{\mathrm{SL}(2)}$ is proper, and the quotient space $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ is Hausdorff.*

(ii) *Assume that Γ is neat. Then the quotient $\Gamma\backslash D^*_{\mathrm{SL}(2)}$ belongs to $\mathcal{B}_{\mathbf{R}}(\log)$, and the projection $D^*_{\mathrm{SL}(2)} \to \Gamma\backslash D^*_{\mathrm{SL}(2)}$ is a local isomorphism of objects of $\mathcal{B}_{\mathbf{R}}(\log)$.*

§8. Application to Hodge metrics at the boundary of $D^I_{\mathrm{SL}(2)}$

We apply our present result to norm estimate.

8.1. Let $F \in D$. For $c > 0$, we define a Hermitian form

$$(\ ,\ )_{F,c} \ :\ H_{0,\mathbf{C}} \times H_{0,\mathbf{C}} \to \mathbf{C}$$

as follows.

For each $w \in \mathbf{Z}$, let

$$(\ ,\ )_{F(\mathrm{gr}^W_w)} : \mathrm{gr}^W_{w,\mathbf{C}} \times \mathrm{gr}^W_{w,\mathbf{C}} \to \mathbf{C}$$

be the Hodge metric on $\mathrm{gr}^W_{w,\mathbf{C}}$ defined by $\langle\ ,\ \rangle_w$ and $F(\mathrm{gr}^W_w)$ (cf. [KNU2] 1.3 (2)). For $v \in H_{0,\mathbf{C}}$ and for $w \in \mathbf{Z}$, let $v_{w,F}$ be the image in $\mathrm{gr}^W_{w,\mathbf{C}}$ of the w-component of v with respect to the canonical splitting of W associated to F. Define

$$(v, v')_{F,c} = \textstyle\sum_{w\in\mathbf{Z}} c^w (v_{w,F}, v'_{w,F})_{F(\mathrm{gr}^W_w)} \quad (v, v' \in H_{0,\mathbf{C}}).$$

Proposition 8.2. *Let Ψ be an admissible set of weight filtrations on $H_{0,\mathbf{R}}$ (7.1). Let β be a distance to Ψ-boundary (7.3). Assume $W \notin \Psi$ (resp. $W \in \Psi$). For each $W' \in \Psi$, let $\beta_{W'} : D \to \mathbf{R}_{>0}$ (resp. $D_{\mathrm{nspl}} \to \mathbf{R}_{>0}$) be the W'-component of β. For $p \in D$, let*

$$(\ ,\ )_{p,\beta} := (\ ,\ )_{p,c} \quad \text{with } c = \textstyle\prod_{W'\in\Psi} \beta_{W'}(p)^{-2}.$$

Let $m : \Psi \to \mathbf{Z}$ be a map, let $V = V_m = \bigcap_{W'\in\Psi} W'_{m(W'),\mathbf{C}}$, and let $\mathrm{Her}(V)$ be the space of all Hermitian forms on V.

Let $(\ ,\ )_{p,\beta,m} \in \mathrm{Her}(V)$ be the restriction of $\prod_{W'\in\Psi} \beta_{W'}(p)^{2m(W')}(\ ,\ )_{p,\beta}$ to V.

(i) *The real analytic map $f : D$ (resp. D_{nspl}) $\to \mathrm{Her}(V)$, $p \mapsto (\ ,\ )_{p,\beta,m}$, extends to a real analytic map $f : D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) $\to \mathrm{Her}(V)$.*

(ii) *For a point $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ (resp. $p \in D^I_{\mathrm{SL}(2)}(\Psi)_{\mathrm{nspl}}$) such that Ψ is the set of weight filtrations associated to p, the limit of $(\ ,\ )_{p,\beta,m}$ at p induces a positive definite Hermitian form on the quotient space*

$$V/(\textstyle\sum_{m'<m} \bigcap_{W'\in\Psi} W'_{m'(W'),\mathbf{C}}),$$

where $m' < m$ means $m'(W') \le m(W')$ for all $W' \in \Psi$ and $m' \neq m$.

§9. Examples

9.1. We describe what kind of SL(2)-orbits of positive rank exist in Examples I–IV, V_3. We consider only an SL(2)-orbit in r variables of rank r, hence $J = \{1, \dots, r\}$ in the following.

Example I. Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, let W be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1$ (resp. $j = 2$), let e'_j be the image of e_j in gr^W_{-2} (resp. gr^W_0). Let $\langle e'_2, e'_2\rangle_0 = 1$, $\langle e'_1, e'_1\rangle_{-2} = 1$, and let $h^{0,0} = h^{-1,-1} = 1$, $h^{p,q} = 0$ for all the other (p,q).

Any SL(2)-orbit of rank > 0 is of rank 1. An SL(2)-orbit in one variable of rank 1 is $((\rho_w, \varphi_w)_w, \mathbf{r})$, where ρ_w is the trivial homomorphism from SL(2) onto the unit group $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ and φ_w is the unique map from $\mathbf{P}^1(\mathbf{C})$ onto the one point set $D(\mathrm{gr}^W_w)$, and $\mathbf{r}$ is any element of $D_{\mathrm{nspl}} = \mathbf{C} \smallsetminus \mathbf{R}$. We have $W^{(1)} = W$.

Example II. Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, let

$$W_{-2} = 0 \subset W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1, 2$ (resp. 3), let e'_j be the image of e_j in gr^W_{-1} (resp. gr^W_0). Let $\langle e'_3, e'_3\rangle_0 = 1$, $\langle e'_2, e'_1\rangle_{-1} = 1$, and let $h^{0,0} = h^{0,-1} = h^{-1,0} = 1$, $h^{p,q} = 0$ for all the other (p,q).

Any SL(2)-orbit of rank > 0 is of rank 1. An SL(2)-orbit in one variable of rank 1 is $((\rho_w, \varphi_w)_w, \mathbf{r})$, where (ρ_w, φ_w) is of rank 0 for $w \neq -1$, and $(\rho_{-1}, \varphi_{-1})$ is of rank 1. An example of such SL(2)-orbit is given by $\rho_{-1} = \mathrm{id} : \mathrm{SL}(2, \mathbf{C}) \xrightarrow{=} G_{\mathbf{C}} = \mathrm{SL}(2, \mathbf{C})$, $\varphi_{-1} = \mathrm{id} : \mathbf{P}^1(\mathbf{C}) \xrightarrow{=} \check{D} = \mathbf{P}^1(\mathbf{C})$, and $\mathbf{r} = \mathbf{r}(i, z) \in D$ for $z \in \mathbf{C}$ defined by

$$\mathbf{r}^1 = 0 \subset \mathbf{r}^0 = \mathbf{C}(ie_1 + e_2) + \mathbf{C}(ze_1 + e_3) \subset \mathbf{r}^{-1} = H_{0,\mathbf{C}}.$$

For this SL(2)-orbit, $W^{(1)}$ is given by

$$W^{(1)}_{-3} = 0 \subset W^{(1)}_{-2} = W^{(1)}_{-1} = \mathbf{R}e_1 \subset W^{(1)}_0 = H_{0,\mathbf{R}}.$$

Example III. Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, let

$$W_{-4} = 0 \subset W_{-3} = W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1, 2$ (resp. 3), let e'_j be the image of e_j in gr^W_{-3} (resp. gr^W_0). Let $\langle e'_3, e'_3\rangle_0 = 1$, $\langle e'_2, e'_1\rangle_{-3} = 1$, and let $h^{0,0} = h^{-1,-2} = h^{-2,-1} = 1$, $h^{p,q} = 0$ for all the other (p,q).

There are three cases for SL(2)-orbits in r variables of rank $r > 0$. For any of them, (ρ_w, φ_w) is of rank 0 unless $w = -3$.

Case 1. $r = 1$ and $(\rho_{-3}, \varphi_{-3})$ is of rank 1. An example of such SL(2)-orbit is given as follows. $(\rho_{-3}, \varphi_{-3}) = (\rho_{-1}, \varphi_{-1}(1))$ where $(\rho_{-1}, \varphi_{-1})$ is in the example in Example

II (we identify $\check{D}(\mathrm{gr}^W_{-3})$ with $\mathbf{P}^1(\mathbf{C})$ via the Tate twist), and $\mathbf{r} = \mathbf{r}(i, z, i)$ for $z \in \mathbf{C}$ is defined by

$$\mathbf{r}^1 = 0 \subset \mathbf{r}^0 = \mathbf{C}(ze_1 + ie_2 + e_3) \subset \mathbf{r}^{-1} = \mathbf{r}^0 + \mathbf{C}(ie_1 + e_2) \subset \mathbf{r}^{-2} = H_{0,\mathbf{C}}.$$

For this SL(2)-orbit,

$$W^{(1)}_{-5} = 0 \subset W^{(1)}_{-4} = W^{(1)}_{-3} = \mathbf{R}e_1 \subset W^{(1)}_{-2} = W^{(1)}_{-1} = W^{(1)}_{-3} + \mathbf{R}e_2 \subset W^{(1)}_0 = H_{0,\mathbf{R}}.$$

Case 2. $r = 1$ and $(\rho_{-3}, \varphi_{-3})$ is of rank 0. An example of such SL(2)-orbit is given as follows. ρ_{-3} is a trivial homomorphism onto $\{1\}$, φ_{-3} is a constant map with value $i \in \mathfrak{h} = D(\mathrm{gr}^W_{-3})$, and $\mathbf{r} = \mathbf{r}(i, z_1, z_2)$ for $(z_1, z_2) \in \mathbf{C}^2 \smallsetminus \mathbf{R}^2$ is the one replaced $\mathbf{r}^0 = \mathbf{C}(ze_1 + ie_2 + e_3)$ by $\mathbf{r}^0 = \mathbf{C}(z_1e_1 + z_2e_2 + e_3)$ in $\mathbf{r}$ in Case 1. For this SL(2)-orbit, $W^{(1)} = W$.

Case 3. $r = 2$ and $(\rho_{-3}, \varphi_{-3})$ is of rank 1. $\rho_{-3} : \mathrm{SL}(2, \mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}^W_{-3}) = \mathrm{SL}(2, \mathbf{C})$ factors through the second projection onto $\mathrm{SL}(2, \mathbf{C})$, and $\varphi_{-3} : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}^W_{-3}) = \mathbf{P}^1(\mathbf{C})$ factors through the second projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such SL(2)-orbit is given as follows. $\rho_{-3}(g_1, g_2) = g_2$, $\varphi_{-3}(p_1, p_2) = p_2$, and $\mathbf{r} = F(i, z_1, z_2)$ for $(z_1, z_2) \in \mathbf{C}^2 \smallsetminus \mathbf{R}^2$ is as in Case 2. For this SL(2)-orbit, $W^{(1)} = W$ and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 1.

Example IV. Let $H_0 = \mathbf{Z}^4 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$, let

$$W_{-3} = 0 \subset W_{-2} = \mathbf{R}e_1 \subset W_{-1} = W_{-2} + \mathbf{R}e_2 + \mathbf{R}e_3 \subset W_0 = H_{0,\mathbf{R}}.$$

For $j = 1$ (resp. $2, 3$, resp. 4), let e'_j be the image of e_j in gr^W_{-2} (resp. gr^W_{-1}, resp. gr^W_0). Let $\langle e'_4, e'_4\rangle_0 = 1$, $\langle e'_1, e'_1\rangle_{-2} = 1$, and $\langle e'_3, e'_2\rangle_{-1} = 1$, and let $h^{0,0} = h^{0,-1} = h^{-1,0} = h^{-1,-1} = 1$, $h^{p,q} = 0$ for all the other (p, q).

There are three cases for SL(2)-orbits in r variables of rank $r > 0$. For any of them, (ρ_w, φ_w) is of rank 0 unless $w = -1$.

Case 1. $r = 1$ and $(\rho_{-1}, \varphi_{-1})$ is of rank 1. An example of such SL(2)-orbit is given as follows. $(\rho_{-1}, \varphi_{-1})$ is as in Example II, and $\mathbf{r} = \mathbf{r}(i, z_1, z_2, z_3)$ for $z_1, z_2, z_3 \in \mathbf{C}$ is defined by

$$\mathbf{r}^1 = 0 \subset \mathbf{r}^0 = \mathbf{C}(z_1e_1 + ie_2 + e_3) + \mathbf{C}(z_2e_1 + z_3e_2 + e_4) \subset \mathbf{r}^{-1} = H_{0,\mathbf{C}}.$$

For this SL(2)-orbit,

$$W^{(1)}_{-3} = 0 \subset W^{(1)}_{-2} = W^{(1)}_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W^{(1)}_0 = H_{0,\mathbf{R}}.$$

Case 2. $r = 1$ and $(\rho_{-1}, \varphi_{-1})$ is of rank 0. An example of such SL(2)-orbit is given as follows. ρ_{-1} is a trivial homomorphism onto $\{1\}$, φ_{-1} is a constant map with value $i \in \mathfrak{h} = D(\mathrm{gr}^W_{-1})$, and $\mathbf{r} = F(i, z_1, z_2, z_3)$ with $\mathrm{Im}(z_2) \neq \mathrm{Im}(z_1)\mathrm{Im}(z_3)$ (the last condition says $\mathbf{r}(i, z_1, z_2, z_3) \in D_{\mathrm{nspl}}$). For this SL(2)-orbit, $W^{(1)} = W$.

Case 3. $r = 2$ and $(\rho_{-1}, \varphi_{-1})$ is of rank 1. $\rho_{-1} : \mathrm{SL}(2, \mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}^W_{-1})$ factors through the second projection onto $\mathrm{SL}(2, \mathbf{C})$, and $\varphi_{-1} : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}^W_{-1}) = \mathbf{P}^1(\mathbf{C})$ factors through the second projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such SL(2)-orbit is given as follows. $\rho_{-1}(g_1, g_2) = g_2$, $\varphi_{-1}(p_1, p_2) = p_2$, and $\mathbf{r} = \mathbf{r}(i, z_1, z_2, z_3)$ with $\mathrm{Im}(z_2) \neq \mathrm{Im}(z_1)\mathrm{Im}(z_3)$. For this SL(2)-orbit, $W^{(1)} = W$ and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 1.

Example $\mathbf{V_3}$. Let $H_0 = \mathbf{Z}^5 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4 + \mathbf{Z}e_5$, let

$$W_{-1} = 0 \subset W_0 = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3 \subset W_1 = H_{0,\mathbf{R}}.$$

For $j = 1, 2, 3$ (resp. 4, 5), let e'_j be the image of e_j in gr^W_0 (resp. gr^W_1). Let $\langle e'_5, e'_4 \rangle_1 = 1$, $\langle e'_1, e'_3 \rangle_0 = 2$, $\langle e'_2, e'_2 \rangle_0 = -1$, and $\langle e'_j, e'_k \rangle_0 = 0$ $(j + k \neq 4,\ 1 \leq j,\ k \leq 3)$, and let $h^{1,-1} = h^{0,0} = h^{-1,1} = h^{1,0} = h^{0,1} = 1$, and $h^{p,q} = 0$ for all the other (p, q).

There are five cases for SL(2)-orbits in r variables of rank $r > 0$. For any of them, (ρ_w, φ_w) is of rank 0 if $w \notin \{0, 1\}$.

Case 1 (resp. *Case* 2). $r = 1$ and (ρ_0, φ_0) is of rank 1 (resp. 0), and (ρ_1, φ_1) is of rank 0 (resp. 1). An example of such SL(2)-orbit is given as follows. (ρ_0, φ_0) (resp. (ρ_1, φ_1)) is $(\mathrm{Sym}^2(\rho), \mathrm{Sym}^2(\varphi)(-1))$ (resp. $(\rho, \varphi(-1))$) for the standard (ρ, φ) (i.e., $(\rho_{-1}, \varphi_{-1})$ in Example II), where (-1) means the Tate twist, and $\mathbf{r} = \mathbf{r}(i, i, z_1, z_2, z_3)$ for $z_1, z_2, z_3 \in \mathbf{C}$ is defined by

$$\begin{aligned} \mathbf{r}^2 = 0 \subset \mathbf{r}^1 &= \mathbf{C}(-e_1 + 2ie_2 + e_3) + \mathbf{C}(z_1e_1 + z_2e_2 + ie_4 + e_5), \\ &\subset \mathbf{r}^0 = \mathbf{r}^1 + \mathbf{C}(ie_1 + e_2) + \mathbf{C}(z_3e_1 + e_4) \subset \mathbf{r}^{-1} = H_{0,\mathbf{C}}. \end{aligned}$$

For this SL(2)-orbit,

$$\begin{aligned} W^{(1)}_{-3} = 0 \subset W^{(1)}_{-2} = W^{(1)}_{-1} = \mathbf{R}e_1 \subset W^{(1)}_0 &= W^{(1)}_{-1} + \mathbf{R}e_2 \\ &\subset W^{(1)}_1 = W^{(1)}_0 + \mathbf{R}e_4 + \mathbf{R}e_5 \subset W^{(1)}_2 = H_{0,\mathbf{R}} \end{aligned}$$

$$(\text{resp. } W^{(1)}_{-1} = 0 \subset W^{(1)}_0 = W^{(1)}_1 = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3 + \mathbf{R}e_4 \subset W^{(1)}_2 = H_{0,\mathbf{R}}).$$

Case 3. $r = 1$, and both (ρ_0, φ_0) and (ρ_1, φ_1) are of rank 1. An example of such SL(2)-orbit is given as follows. $\rho_0 = \mathrm{Sym}^2(\rho)$, $\varphi_0 = \mathrm{Sym}^2(\varphi)(-1)$, $\rho_1 = \rho$, $\varphi_1 = \varphi(-1)$ for the standard (ρ, φ) (i.e., $(\rho_{-1}, \varphi_{-1})$ in Example II), and $\mathbf{r} = \mathbf{r}(i, i, z_1, z_2, z_3)$ for $z_1, z_2, z_3 \in \mathbf{C}$. For this SL(2)-orbit,

$$W^{(1)}_{-3} = 0 \subset W^{(1)}_{-2} = W^{(1)}_{-1} = \mathbf{R}e_1 \subset W^{(1)}_0 = W^{(1)}_1 = W^{(1)}_{-1} + \mathbf{R}e_2 + \mathbf{R}e_4 \subset W^{(1)}_2 = H_{0,\mathbf{R}}.$$

Case 4 (resp. *Case* 5). $r = 2$, both (ρ_0, φ_0) and (ρ_1, φ_1) are of rank 1, $\rho_0 : \mathrm{SL}(2, \mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}^W_0)$ factors through the first (resp. second) projection onto $\mathrm{SL}(2, \mathbf{C})$, $\varphi_0 : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}^W_0)$ factors through the first (resp. second) projection onto $\mathbf{P}^1(\mathbf{C})$, $\rho_1 : \mathrm{SL}(2, \mathbf{C})^2 \to G_{\mathbf{C}}(\mathrm{gr}^W_1)$ factors through the second (resp. first) projection onto

$\mathrm{SL}(2,\mathbf{C})$, $\varphi_1 : \mathbf{P}^1(\mathbf{C})^2 \to \check{D}(\mathrm{gr}_1^W)$ factors through the second (resp. first) projection onto $\mathbf{P}^1(\mathbf{C})$. An example of such SL(2)-orbit is given as follows. For $j = 1$ (resp. 2), $\rho_0(g_1, g_2) = \mathrm{Sym}^2(g_j)$, $\varphi_0(p_1, p_2) = p_j \in \mathbf{P}^1(\mathbf{C}) \simeq \check{D}(\mathrm{gr}_0^W)$, $\rho_1(g_1, g_2) = g_{3-j}$, $\varphi_1(p_1, p_2) = p_{3-j}(-1) \in \mathbf{P}^1(\mathbf{C}) \simeq \check{D}(\mathrm{gr}_1^W)$, and $\mathbf{r} = \mathbf{r}(i, i, z_1, z_2, z_3)$ with $z_1, z_2, z_3 \in \mathbf{C}$. For this SL(2)-orbit, $W^{(1)}$ is the $W^{(1)}$ in the example in Case 1 (resp. Case 2) and $W^{(2)}$ is the $W^{(1)}$ in the example in Case 3.

9.2. By Proposition 7.8, $D^I_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$ for Examples I–IV. We describe the structure of the open set $D^{II}_{\mathrm{SL}(2)}(\Phi)$ of $D^{II}_{\mathrm{SL}(2)}$ for some $\Phi \in \overline{\mathcal{W}}$.

Let $\bar{\mathfrak{h}} = \{x + iy \mid x, y \in \mathbf{R}, 0 < y \le \infty\} \supset \mathfrak{h}$. We regard $\bar{\mathfrak{h}}$ as an object of $\mathcal{B}_{\mathbf{R}}(\log)$ via $\bar{\mathfrak{h}} \simeq \mathbf{R}_{\ge 0} \times \mathbf{R}$, $x + iy \mapsto (1/\sqrt{y}, x)$.

Example I. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$\begin{array}{ccc} D & \simeq & \mathrm{spl}(W) \times L \\ \cap & & \cap \\ D_{\mathrm{SL}(2)} & \simeq & \mathrm{spl}(W) \times \bar{L}. \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = D(\mathrm{gr}^W)$ which is just a one point set, $L \simeq \mathbf{R}$ with weight -2, and $\bar{L}$ is isomorphic to the interval $[-\infty, \infty]$ endowed with the real analytic structure as in [KNU2], 7.5, with $w = -2$ which contains $\mathbf{R} = L$ in the natural way.

Example II. Let $\Phi = \bar{\Psi}$ for $\Psi = W^{(1)}$ in 9.1 Example II. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$\begin{array}{ccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \\ \cap & & \cap \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}}. \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^2$. In this diagram, the lower isomorphism is induced by the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Phi) \simeq \bar{\mathfrak{h}}$.

The specific examples of SL(2)-orbits of rank 1 in 9.1 Example II have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}}$ are $i\infty$.

Example III. Let $\Phi = \bar{\Psi}$ for $\Psi = W^{(1)}$ in 9.1 Example III. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$\begin{array}{ccccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \times L & & (s,\ x + iy,\ (d_1,\ d_2)) \\ \cap & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} & & (s,\ x + iy,\ (y^{-2}d_1,\ y^{-1}d_2)). \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^2$, $L \simeq \mathbf{R}^2$ with weight -3, and $(d_1, d_2) \in \mathbf{R}^2 = L$. In this diagram, the lower isomorphism is induced by the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Phi) \simeq \bar{\mathfrak{h}}$, and the following morphism $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$. It is induced by $\nu_{\alpha,\beta}$ (7.4 (ii)), where $\alpha_{-3} : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_{-3})$ is defined by $\alpha_{-3}(t)e_1 = t^{-4}e_1, \alpha_{-3}(t)e_2 = t^{-2}e_2$, and $\beta : D(\mathrm{gr}^W_{-3}) = \mathfrak{h} \to \mathbf{R}_{>0}$ is the distance to Φ-boundary defined by $x + iy \mapsto 1/\sqrt{y}$. Note that the right vertical arrow is *not* the evident map, as indicated.

The SL(2)-orbits in 9.1 Example III, Case 1 (resp. Case 2, resp. Case 3) have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}} \times \bar{L}$ belong to $\{i\infty\} \times L$ (resp. $\{i\} \times (\bar{L} \smallsetminus L)$, resp. $\{i\infty\} \times (\bar{L} \smallsetminus L)$).

Example IV. Let $\Phi = \bar{\Psi}$ for $\Psi = W^{(1)}$ in 9.1 Example IV. We have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$\begin{array}{ccccc} D & \simeq & \mathrm{spl}(W) \times \mathfrak{h} \times L & & (s,\, x+iy,\, d) \\ \cap & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} & & (s,\, x+iy,\, y^{-1}d). \end{array}$$

Here $\mathrm{spl}(W) \simeq \mathbf{R}^5$, $L \simeq \mathbf{R}$ with weight -2, and $d \in \mathbf{R} = L$. In this diagram, the lower isomorphism is induced from the canonical morphisms $D^{II}_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ and $D^{II}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(\Phi) \simeq \bar{\mathfrak{h}}$, and the following morphism $D^{II}_{\mathrm{SL}(2)}(\Phi) \to \bar{L}$. It is induced by $\nu_{\alpha,\beta}$ (7.4 (ii)), where $\alpha_{-1} : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}(\mathrm{gr}^W_{-1})$ is defined by

$$\alpha_{-1}(t)e'_2 = t^{-2}e'_2, \quad \alpha_{-1}(t)e'_3 = e'_3 \quad \text{with } e'_j = e_j \bmod W_{-2} \quad (j = 2, 3),$$

and $\beta : D(\mathrm{gr}^W_{-1}) = \mathfrak{h} \to \mathbf{R}_{>0}$ is the distance to Φ-boundary defined by $x + iy \mapsto 1/\sqrt{y}$. Note that the right vertical arrow is *not* the inclusion map, as indicated.

The SL(2)-orbits in 9.1 Example IV, Case 1 (resp. Case 2, resp. Case 3) have classes in $D^{II}_{\mathrm{SL}(2)}(\Phi)$ whose images in $\bar{\mathfrak{h}} \times \bar{L}$ belong to $\{i\infty\} \times L$ (resp. $\{i\} \times (\bar{L} \smallsetminus L)$, resp. $\{i\infty\} \times (\bar{L} \smallsetminus L)$).

9.3. By Proposition 7.8, $D^{I}_{\mathrm{SL}(2)} \neq D^{II}_{\mathrm{SL}(2)}$ for Example V_3. For this example, we describe the structures of the open sets $D^{I}_{\mathrm{SL}(2)}(\Psi)$ of $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}(\bar{\Psi})$ of $D^{II}_{\mathrm{SL}(2)}$ for $\Psi = \{W^{(1)}\} \in \mathcal{W}$ with $W^{(1)}$ in 9.1, V_3, Case 1.

For $j = 1, 2, 3$, let $A_j = \mathrm{Hom}_{\mathbf{R}}(\mathrm{gr}^W_1, \mathbf{R}e_j)$. We have an isomorphism of real analytic manifolds

$$\mathrm{spl}(W) \xrightarrow{\sim} \textstyle\prod_{j=1}^3 A_j, \quad s \mapsto (a_j)_{1 \le j \le 3},$$

$$\text{where} \quad s(v) \equiv \textstyle\sum_{j=1}^3 a_j(v) \bmod \mathbf{R}e_4 + \mathbf{R}e_5 \quad \text{for } v \in \mathrm{gr}^W_1\,.$$

Let

$$(A_3 \times \bar{\mathfrak{h}}^{\pm})' := \{(v, x+iy) \in A_3 \times \bar{\mathfrak{h}}^{\pm} \mid v = 0 \text{ if } y = \pm\infty\} \subset A_3 \times \bar{\mathfrak{h}}^{\pm}.$$

Then we have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$(1)\qquad \begin{array}{ccc} D & \simeq & (\prod_{j=1}^{3} A_j) \times \mathfrak{h}^{\pm} \times \mathfrak{h} \\ \cap & & \cap \\ D^{II}_{\mathrm{SL}(2)}(\bar{\Psi}) & \simeq & (\prod_{j=1}^{3} A_j) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}. \end{array}$$

Here $\bar{\mathfrak{h}}^{\pm}$ is the disjoint union of $\bar{\mathfrak{h}}^{+} = \bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^{-} = \{x+iy \mid x \in \mathbf{R},\, 0 > y \geq -\infty\}$. In this diagram, the upper isomorphism is induced by the isomorphism in §2 and the above isomorphism $\mathrm{spl}(W) \simeq \prod_{j=1}^{3} A_j$. On the other hand, we have a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$(2)\qquad \begin{array}{ccccc} D & \simeq & (\prod_{j=1}^{3} A_j) \times \mathfrak{h}^{\pm} \times \mathfrak{h} & \ni & (a_1, a_2, a_3, x+iy, \tau) \\ \cap & & \downarrow & & \downarrow \\ D^{I}_{\mathrm{SL}(2)}(\Psi) & \simeq & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & \ni & (a_1, a_2, |y|^{1/2}a_3, x+iy, \tau). \end{array}$$

In this diagram (2), the upper isomorphism is the same as in diagram (1). The lower isomorphism is induced from the canonical morphisms $D^{I}_{\mathrm{SL}(2)} \to \mathrm{spl}(W) \to A_1 \times A_2$ and $D^{I}_{\mathrm{SL}(2)}(\Psi) \to D^{II}_{\mathrm{SL}(2)}(\bar{\Psi}) \to \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$, and the following morphism $D^{I}_{\mathrm{SL}(2)}(\Psi) \to A_3$. It is the composite

$$D^{I}_{\mathrm{SL}(2)}(\Psi) \xrightarrow{\text{by } \nu_{\alpha,\beta}} D \xrightarrow{\mathrm{spl}_W} \mathrm{spl}(W) \simeq \prod_{j=1}^{3} A_j \to A_3,$$

where $\nu_{\alpha,\beta}$ is the morphism described in 7.4 (i). Here $\alpha : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the splitting of Ψ defined by $\alpha(t)e_1 = t^{-2}e_1$, $\alpha(t)e_2 = e_2$, $\alpha(t)e_3 = t^2e_3$, $\alpha(t)e_4 = te_4$, $\alpha(t)e_5 = te_5$, and $\beta : D \to \mathbf{R}_{>0}$ is the distance to Ψ-boundary defined as the composite $D \to D(\mathrm{gr}^W_0) \simeq \mathfrak{h}^{\pm} \to \mathbf{R}_{>0}$, where the last arrow is $x+iy \mapsto 1/\sqrt{|y|}$.

Note that the right vertical arrow of the commutative diagram (2) is *not* the inclusion map, as indicated.

The lower isomorphisms in two commutative diagrams (1) and (2) form a commutative diagram in $\mathcal{B}_{\mathbf{R}}(\log)$

$$(3)\qquad \begin{array}{ccccc} D^{I}_{\mathrm{SL}(2)}(\Psi) & \simeq & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & \ni & (a_1, a_2, a_3, x+iy, \tau) \\ \downarrow & & \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\bar{\Psi}) & \simeq & (\prod_{j=1}^{3} A_j) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h} & \ni & (a_1, a_2, |y|^{-1/2}a_3, x+iy, \tau). \end{array}$$

Here in diagram (3), the left vertical arrow is the inclusion map. The right vertical arrow is *not* the evident map, as indicated.

The SL(2)-orbits in 9.1, V_3, Case 1 have classes in $D^{I}_{\mathrm{SL}(2)}(\Psi)$ whose images in $\bar{\mathfrak{h}}^{\pm} \times \mathfrak{h}$ are $(i\infty, i)$.

9.4. We describe that in Example V_3 the norm estimate is not continuous on $D^{II}_{\mathrm{SL}(2)}$. Let Ψ and $\Phi := \bar{\Psi}$ be as in 9.3. Fix $u, v \in \mathbf{C}e_4 + \mathbf{C}e_5$. Let $\beta : D \to \mathbf{R}_{>0}$ be the distance to Ψ-boundary which appears in 9.3.

As in Proposition 8.2, the map

$$f : D \to \mathbf{C}, \quad p \mapsto \beta(p)^2(u,v)_{p,\beta},$$

extends to a real analytic function $f : D^I_{\mathrm{SL}(2)}(\Psi) \to \mathbf{C}$. This is understood by the description of $D^I_{\mathrm{SL}(2)}(\Psi)$ given in 9.3:

$$\begin{array}{ccccc} (a_1,\, a_2,\, (a_3,\, x+iy),\, \tau) & \in & A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} & \simeq & D^I_{\mathrm{SL}(2)}(\Psi) \\ \downarrow & & \downarrow & & \downarrow \\ (a_1,\, a_2,\, |y|^{-1/2}a_3,\, x+iy,\, \tau) & \in & (\prod_{j=1}^3 A_j) \times \bar{\mathfrak{h}}^{\pm} \times \mathfrak{h} & \simeq & D^{II}_{\mathrm{SL}(2)}(\Phi). \end{array}$$

The composite $A_1 \times A_2 \times (A_3 \times \bar{\mathfrak{h}}^{\pm})' \times \mathfrak{h} \simeq D^I_{\mathrm{SL}(2)}(\Psi) \xrightarrow{f} \mathbf{C}$ sends $(a_1,\, a_2,\, (a_3,\, x+iy),\, \tau)$ to

$$\begin{aligned} &(|y|^{-3/2}a_1(u) + |y|^{-1/2}a_2(u) + |y|^{1/2}a_3(u), \\ &|y|^{-3/2}a_1(v) + |y|^{-1/2}a_2(v) + |y|^{1/2}a_3(v))_{0,(x+iy)/|y|} + (u,v)_{1,\tau}. \end{aligned}$$

Here $(\ ,\ )_{0,(x+iy)/|y|}$ is the Hodge metric on $\mathrm{gr}^W_{0,\mathbf{C}}$ associated to $(x+iy)/|y| \in \mathfrak{h}^{\pm} = D(\mathrm{gr}^W_0)$ and $(\ ,\ )_{1,\tau}$ is the Hodge metric on $\mathrm{gr}^W_{1,\mathbf{C}}$ associated to $\tau \in \mathfrak{h} = D(\mathrm{gr}^W_1)$.

However, f is not necessarily continuous for the topology of $D^{II}_{\mathrm{SL}(2)}$, as is seen from the above commutative diagram. In fact, let $a_3 \in A_3$ and assume $(a_3(u), a_3(v))_{0,i} \neq 0$. Let $p(y) \in D$ for $y > 0$ (resp. $q \in D^{II}_{\mathrm{SL}(2)}$) be the image of the point $(0, 0, (a_3, iy), i)$ (resp. $(0, 0, (0, i\infty), i)$) of the left upper space in the diagram. Then, as $y \to \infty$, $p(y)$ converges to q in $D^{II}_{\mathrm{SL}(2)}$ because the image of $(0, 0, (a_3, iy), i)$ under the left vertical arrow is $(0, 0, y^{-1/2}a_3, iy, i)$. But $f(p(y)) = (y^{1/2}a_3(u), y^{1/2}a_3(v))_{0,i} + (u,v)_{1,i}$ diverges.

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